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Effect of Rotation on Rayleigh-Bénard-Marangoni Convection in a Relatively Hotter or Cooler Layer of Liquid with Insulating Boundaries

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Abstract

The effect of uniform rotation on the onset of steady Rayleigh-Bénard-Marangoni convection in a horizontal layer of liquid is investigated, using the modified linear stability analysis. The upper boundary surface of the liquid layer is free where surface tension gradients arise on account of variation of temperature and the lower boundary surface is rigid, each subject to constant heat flux condition. Both mechanisms namely, surface tension and buoyancy causing instability are taken into account. The Galerkin method is used to obtain the eigenvalue equation which is then computed numerically. Results of this analysis indicate that the critical eigenvalues in the presence of a uniform rotation are greater in a relatively hotter layer of liquid than a cooler one under identical conditions otherwise. The asymptotic behaviour of both the critical Rayleigh and Marangoni numbers for large values of the Taylor number are also obtained. During the course of this analysis, we also correct the inaccuracies in the work of earlier authors.

Keywords: Buoyancy, Convection, Insulating, Linear stability, Rotation, Steady, Surface tension.

1. Introduction

The rotation is known to have stabilizing effect, a fact that has already been established by Chandrasekhar [1] for the buoyancy driven convection, and by Vidal and Acrivos [2] for the surface tension driven convection. McConaghy and Finlayson [3] re-examined the problem considered by Vidal and Acrivos [2] on the possibility of oscillatory convection in a rotating fluid layer. Namikawa [4] studied the effect of rotation on the steady onset of combined surface tension and buoyancy driven convection while Kaddame and Lebon [5, 6] investigated the onset of steady and oscillatory Bénard Marangoni convection with rotation. Recently, the effect of uniform rotation on the onset of combined surface tension and buoyancy driven convection has been studied by Gupta and Dhiman [7] for thermally conducting case of the rigid lower boundary and insulating free upper boundary, using the modified linear stability analysis of Banerjee et al [8].

In this paper, we investigate the effect of uniform rotation on the onset of combined surface tension and buoyancy driven thermal convection in a relatively hotter or cooler layer of liquid in which the heat flux across each boundary is kept constant, using the modified linear stability analysis. The present analysis extends the work of Gupta and Surya [9] to include the effect of uniform rotation. The Galerkin method is used to obtain the eigenvalue equation analytically. The numerical results obtained for a wide range of the parameters involved are presented. The results of this analysis indicate that the rotation suppresses convection more effectively in a relatively hotter layer of liquid than the cooler one, irrespective of whether the two mechanisms namely, surface tension and buoyancy causing instability act individually or simultaneously. The two mechanisms causing instability are found to reinforce each other and are perfectly coupled in the absence of rotation. However, for increasing speed of rotation, it is found that the two mechanisms causing instability no longer remain perfectly coupled that is, coupling between them becomes less tight. It is interesting to note that the critical wave number at the onset of convection is found to be zero up to a certain threshold speed of rotation. The asymptotic behavior of both the critical Rayleigh and Marangoni numbers for large values of the Taylor number are obtained. A detail description of the marginal stability curves showing the influence of the uniform rotation on the onset of convection in a relatively hotter or cooler layer of liquid is also given. During the course of this analysis, we correct the inaccuracies in the numerical results of Friedrich and Rudraiah [11].

2. Formulation of the Problem

We consider an infinite horizontal layer of viscous and incompressible fluid of uniform thickness d heated from below which is kept rotating with a constant angular velocity $\vec{\Omega}$ about an axis parallel to the direction of gravity $\vec{g}$. The lower boundary surface of the layer of liquid is rigid and the upper surface is free non-deformable where surface tension gradients arise on account to variation of temperature with the upper free surface open to ambient air where surface tension gradients arise due to temperature perturbations. We choose a Cartesian coordinate system of axes with the x and y axis in the plane of the lower surface and the z-axis along the vertically upward direction so that the fluid is confined between the planes at $z = 0$ and $z = d$. A temperature gradient is maintained across the layer by maintaining the lower boundary at a constant temperature T_0 and the upper boundary at T_1 ($< T_0$). The surface tension on the upper free surface of the fluid is regarded as a function of temperature only which is given by the simple linear law $\tau = \tau_1 - \sigma(T - T_1)$ where the constant τ_1 is the unperturbed value of τ at the unperturbed surface temperature $T = T_1$ and $-\sigma = (\partial\tau/\partial T)_{T=T_1}$ represents the rate of change of surface tension with temperature, evaluated at temperature T_1, and surface tension being a monotonically decreasing function of temperature, σ is positive. We wish to investigate the effect of the rotation on the onset of convection driven under the joint action of surface tension and buoyancy, in the framework of modified linear stability analysis of Banerjee et al [8]. Following Banerjee et al [8], we can write modified linearized perturbation equations under uniform rotation in the relevant context as

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w = g\alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta + 2\Omega\frac{\partial}{\partial z}\zeta, \tag{2.1}$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the z-component of vorticity.

$$(1 - \alpha_2 T_0)\left(\frac{\partial\theta}{\partial t} - \beta w\right) = \kappa\nabla^2\theta, \tag{2.2}$$

$$\frac{\partial}{\partial t}\zeta = \nu\nabla^2\zeta + 2\Omega\frac{\partial w}{\partial z}. \tag{2.3}$$

Where w is the perturbation velocity θ is the perturbation temperature and ρ is the density of fluid. The kinematic viscosity ν, the thermal diffusivity κ, the gravitational acceleration g, the temperature gradient β which is maintained and are each assumed to be constant. Further, that coefficient α_2 (due to variation in the temperature) is a constant that ranges from 0 to 10^{-4} for the liquid with which we are most concerned, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and t denotes time.

In seeking solutions of the Eqn. (2.1), (2.2) and (2.3), we must satisfy certain boundary conditions. The boundary conditions at the lower rigid and thermally insulating surface $z = 0$ are

$$w=0, \quad \frac{\partial w}{\partial z}=0, \quad \frac{\partial \theta}{\partial z}=0, \quad \zeta=0. \qquad (2.4a, b, c, d)$$

The boundary conditions at the upper free and thermally insulating surface $z = d$ are

$$w=0, \quad \rho\nu \frac{\partial^2 w}{\partial z^2}=\sigma\nabla_1^2\theta, \quad \frac{\partial \theta}{\partial z}=0, \quad \frac{\partial \zeta}{\partial z}=0, \qquad (2.5a, b, c, d)$$

Where $\nabla_1^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

We now suppose that the perturbations $w,\ \theta$ and h_z are of the form

$$[w,\ \theta,\ \zeta]=[w(z),\ \theta(z),\ Z(z)]\exp[i(a_x x + a_y y)+pt],$$

Where $a=\sqrt{a_x^2+a_y^2}$ is the wave number of the disturbance and p is the time constant (which can be complex). We now introduce the non-dimensional quantities using $d,\ d^2/\nu,\ \nu/d$ and $\beta d\nu/\kappa$ as the appropriate scales for length, time, velocity and temperature respectively and putting $a_*=ad,\ w_*=\frac{wd}{\nu},\ \theta_*=\frac{\theta\kappa Ra_*^2}{\beta d\nu},\ p_*=\frac{pd^2}{\nu},\ Z_*=\frac{Zd^2}{\nu}.$

We now let x, y and z stand for co-ordinates in the new units and omitting asterisk for simplicity, Eqn. (2.1)-(2.3) and boundary conditions (2.4a, b, c, d)-(2.5a, b, c, d) can be reduced to the following non-dimensional form

$$\left(D^2-a^2\right)\left(D^2-a^2-p\right)w=\theta+\mathrm{T}^{\frac{1}{2}}DZ, \qquad (2.6)$$

$$\left(D^2-a^2-(1-\alpha_2T_0)pP_r\right)\theta=-Ra^2\left(1-\alpha_2T_0\right)w, \qquad (2.7)$$

$$\left(D^2-a^2-p\right)Z=-\mathrm{T}^{\frac{1}{2}}Dw, \qquad (2.8)$$

$$w=0, \quad Dw=0, \quad D\theta=0, \quad Z=0, \quad \text{at} \quad z=0, \qquad (2.9a, b, c, d)$$

$$w=0, \quad D^2w+\Gamma\theta=0, \quad D\theta=0, \quad DZ=0, \quad \text{at} \quad z=1. \qquad (2.10a, b, c, d)$$

where $\mathrm{T}=4\Omega^2d^4/\nu^2$ is the Taylor number, $R=g\alpha\beta d^4/\kappa\nu$ is the Rayleigh number, $P_r=\nu/\kappa$ is the thermal Prandtl number and the parameter $\Gamma=M/R=\sigma/\rho g\alpha d^2$ with $M=\sigma\beta d^2/\rho\kappa\nu$ as the Marangoni number, characterizes the strength of surface tension relative to buoyancy.

We restrict our analysis to the case when the marginal state is stationary so that the marginal state is characterized by setting $p=0$ in Eqn. (2.6)-(2.8). Thus equations governing the stationary marginal state are obtained as

$$\left(D^2-a^2\right)^2w=\theta+\mathrm{T}^{\frac{1}{2}}DZ, \qquad (2.11)$$

$$(D^2-a^2)\theta=-Ra^2(1-\alpha_2T_0)w, \qquad (2.12)$$

$$\left(D^2-a^2\right)Z=-\mathrm{T}^{\frac{1}{2}}Dw. \qquad (2.13)$$

The Eqn. (2.11)-(2.13) together with boundary conditions (2.9a, b, c, d)-(2.10a, b, c, d) constitute an eigenvalue problem of order eight.

3. Solution of the Problem

The single term Galerkin method is convenient for solving the present problem (Finlayson [10]). Accordingly, the unknown variables $w,\ \theta$ and Z are written as

$$w = Aw_1, \quad \theta = B\theta_1, \quad \text{and} \quad Z = CZ_1, \tag{3.1}$$

in which *A*, *B* and *C* are constants w_1, θ_1 and Z_1 are the trial functions, which are chosen suitably satisfying the boundary conditions (2.9a, b, c, d)-(2.10a, b, c, d). Multiplying Eqn. (2.11) by *w*, Eqn. (2.12) by θ, and Eqn. (2.13) by Z_1, integrating the resulting equations with respect to *z* from 0 to 1 using the boundary conditions (2.9a, b, c, d)-(2.10a, b, c, d). Substituting for *w*, θ and *Z* from (3.1) and eliminating *A*, *B* and *C* from resulting system of equations, we obtain the following eigenvalue equation

$$\begin{vmatrix} \left\langle \left(D^2 w\right)^2 + 2a^2 (Dw)^2 + a^4 (w)^2 \right\rangle & \left[\langle w\theta\rangle - \Gamma Dw(1)\theta(1)\right] & (\mathrm{T})^{\frac{1}{2}} \langle wDZ\rangle \\ Ra^2 (1-\alpha_2 T_0) \langle w\theta\rangle & \left\langle (D\theta)^2 + a^2 (\theta)^2 \right\rangle & 0 \\ (\mathrm{T})^{\frac{1}{2}} \langle Z\,Dw\rangle & 0 & \left\langle (DZ)^2 + a^2 (Z)^2 \right\rangle \end{vmatrix} = 0, \tag{3.2}$$

Where $\langle --\rangle$ denotes integration with respect to *z* from *z* = 0 to 1 and suffixes have been dropped for simplicity while writing the Eq. (3.2). The eigen value Eqn. (3.2) may be put in the following form

$$R = \frac{1}{(1-\alpha_2 T_0) a^2 \left[\left(\langle w\theta\rangle\right)^2 - \Gamma Dw(1)\theta(1)\langle w\theta\rangle \right]} \times \left\{ \left[\left\langle \left(D^2 w\right)^2 + 2a^2 (Dw)^2 + a^4 (w)^2 \right\rangle \left\langle (D\theta)^2 + a^2 (\theta)^2 \right\rangle \right] + \mathrm{T} \left[\frac{\left(\langle wDZ\rangle\right)^2 \left\langle (D\theta)^2 + a^2 (\theta)^2 \right\rangle}{\left\langle (DZ)^2 + a^2 (Z)^2 \right\rangle} \right] \right\}. \tag{3.3}$$

We select the trial function

$$w = z^2 (1-z) \left[\frac{\Gamma}{4} + \frac{1}{24} \left(\frac{3}{2} - z \right) \right], \qquad \theta = 1, \qquad \text{and} \qquad Z = z\left(1 - \frac{z}{2}\right), \tag{3.4}$$

Such that they satisfy all the boundary conditions (2.9a, b, c, d)-(2.10a, b, c, d). It is important to remark here that above choice of the velocity trial function given by (3.4) is found to be useful for cases in which the two mechanisms (buoyancy and surface tension) causing instability act individually or simultaneously. Substitution of trial functions given by (3.4) into the eigen value Eqn. (3.3), we get

$$R\left(\frac{1}{320} + \frac{\Gamma}{48}\right) = \frac{1}{(1-\alpha_2 T_0)} \left\{ 1 + \frac{a^2}{15} \left[\frac{\left(\Gamma + \frac{1}{8}\right)^2 + \frac{1}{448}}{\left(\Gamma + \frac{1}{12}\right)^2 + \frac{1}{180}} \right] + \frac{a^4}{420} \left[\frac{\left(\Gamma + \frac{7}{48}\right)^2 + \frac{5}{6912}}{\left(\Gamma + \frac{1}{12}\right)^2 + \frac{1}{180}} \right] + \frac{\mathrm{T}}{240} \left[\frac{\left(\Gamma + \frac{1}{6}\right)^2}{\left(5 + 2a^2\right) \left[\left(\Gamma + \frac{1}{12}\right)^2 + \frac{1}{180} \right]} \right] \right\}. \tag{3.5}$$

4. Numerical Results and Discussion

The numerical calculations are carried out using the symbolic algebraic package Mathematica, for assigned values of the parameters Γ, $\alpha_2 T_0$ and T. We seek the minimum of R as a function of the wave number a to obtain values of the critical Rayleigh number R_c and corresponding critical wave number. Validation of the computer program is achieved through verification of existing results obtained by Gupta and Surya [9].

The limiting cases of the parameter Γ in the relation (3.5), give rise to the following two cases namely, when buoyancy is the sole agency causing instability and when surface tension is the sole agency causing instability.

Case I When buoyancy is sole agency causing instability

When $\Gamma \to 0$ (or $M \to 0$) implies that in the absence of surface tension effect, buoyancy is the sole agency causing instability. In this case, we obtain R from the eigen value Eqn. (3.5) in terms of a, $\alpha_2 T_0$ and T as

$$R = \frac{320}{(1-\alpha_2 T_0)}\left\{1+\frac{2a^2}{21}+\frac{19a^4}{4536}+\frac{\mathrm{T}}{108\left(5+2a^2\right)}\right\}. \tag{4.1}$$

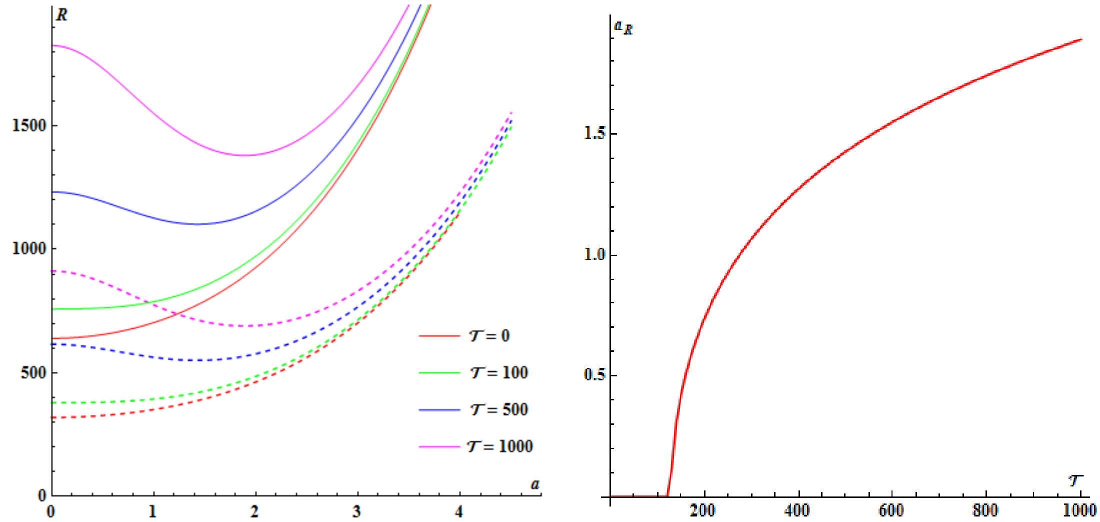
The numerical values of the critical Rayleigh number R_c and corresponding wave number a_R, computed using the relation (4.1), for various values of $\alpha_2 T_0$ and T are presented in Table 1. When $\mathrm{T} = 0$ (in the absence of rotation), from Table 1 we find that values of both R_c and a_R for various values of $\alpha_2 T_0$ agree precisely with corresponding values obtained by Gupta and Surya [9]. For a prescribed value of $\alpha_2 T_0$, an increase in the value of T leads to increased values of both R_c and a_R, indicating that the rotation has stabilizing effect with formation of cells of decreased sizes on the onset of buoyancy driven convection. On the other hand, for a prescribed value of T, an increase in the value of $\alpha_2 T_0$ leads to a greater value of R_c indicating that hotter the liquid layer more the postponement of the onset of instability. It is interesting to note that value of the critical wave number a_R remains unchanged for various values of $\alpha_2 T_0$.

The (a, R) curves corresponding to neutral stability are plotted in Fig. 1(a), using the relation (4.1) for various prescribed values of T when $\alpha_2 T_0$ =0 (dotted curves) and $\alpha_2 T_0$ = 0.5 (thick curves), which shows that the rotation has stabilizing effect on the onset of buoyancy driven convection. Further, Fig. 1(a) also illustrates that a relatively hotter layer of liquid is more stable than the cooler one under the effect of rotation.

Table 1 Values of R_c and a_R for various values of T when $\alpha_2 T_0$ = 0, 0.3 and 0.5.

T	$\alpha_2 T_0 = 0$		$\alpha_2 T_0 = 0.3$		$\alpha_2 T_0 = 0.5$	
	R_c	a_R	R_c	a_R	R_c	a_R
0	320.00	0.00	457.14	0.00	640.00	0.00
1	320.59	0.00	457.99	0.00	641.19	0.00
10	325.93	0.00	465.61	0.00	651.85	0.00
100	379.26	0.00	541.80	0.00	758.52	0.00
130	397.04	0.11	567.19	0.11	794.07	0.11
250	457.43	0.93	653.47	0.93	914.86	0.93
500	550.91	1.43	787.01	1.43	1101.82	1.43
10^3	689.95	1.89	985.64	1.89	1379.90	1.89
10^4	1897.89	3.56	2711.27	3.56	3795.78	3.56
10^6	29248.50	8.76	41783.5	8.76	58496.90	8.76
10^8	592685.0	19.38	846693	19.38	1.19×10^6	19.38

The variation of the critical wave number a_R with Taylor number T at the onset of convection is illustrated in Fig. 1(b). The numerical calculations indicate that the marginal stability curves have minimum of R at $a_R = 0$ up to a certain threshold value of the Taylor number T ($\leq$130 approximately) and R_c occurs at a non-zero value of a_R when T is greater than the threshold.



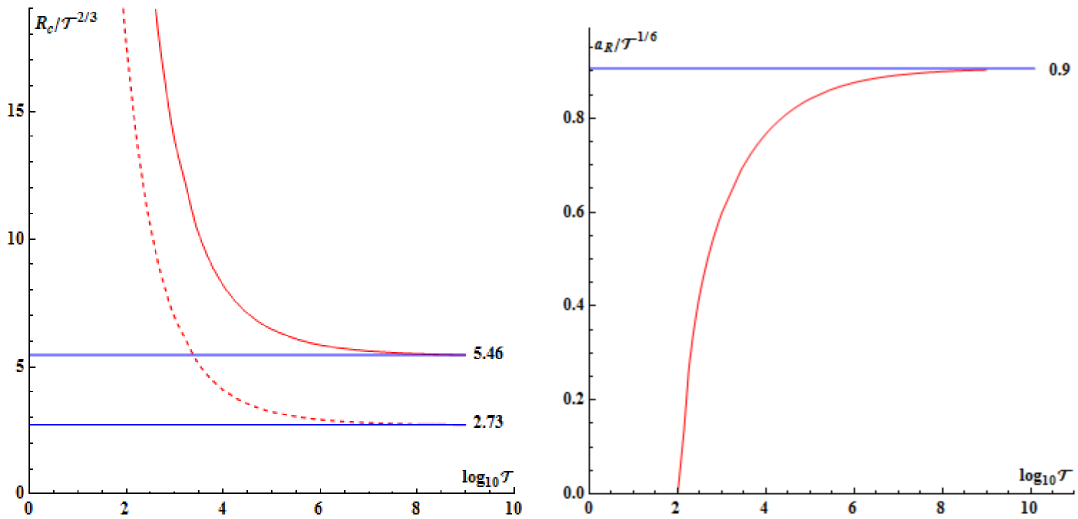
(a) (b)

Figure. 1. (a) Neutral stability curves at the onset of Rayleigh Bènard convection for various values of T when $\alpha_2 T_0 = 0$ (dotted curves) $\alpha_2 T_0 = 0.5$ (thick curves).
(b) Variation of a_R as a function of T .

The asymptotic behavior of R_c and a_R obtained numerically for large value of Taylor number T are given as

$$R_c \approx \frac{2.73}{(1-\alpha_2 T_0)}(\mathrm{T})^{\frac{2}{3}} \text{ and } a_R \approx 0.9(\mathrm{T})^{\frac{1}{6}} \tag{4.2}$$

We find that the asymptotic behavior of R_c crucially depends on both T and $\alpha_2 T_0$, whereas the asymptotic behavior of a_R only depends on T and is independent of $\alpha_2 T_0$.



(a) (b)

Figure 2 Asymptotic results in the limit $\mathrm{T} \to \infty$ plotted as function of $\log_{10}\mathrm{T}$

(a) $R_c/\mathrm{T}^{\,2/3}$ when $\alpha_2T_0 = 0$ (dotted curve) $\alpha_2T_0 = 0.5$ (thick curve). (b) $a_R/\mathrm{T}^{\,1/6}$.

Fig. 2(a) shows numerically calculated values of $R_c/\mathrm{T}^{\,2/3}$ plotted as a function of $\log_{10}\mathrm{T}$ and verifies that $R_c/\mathrm{T}^{\,2/3} \approx 2.73$ and 5.46 respectively when $\alpha_2T_0 = 0$ (dotted curve) and $\alpha_2T_0 = 0.5$ (thick curve), as $\mathrm{T}\to\infty$. Fig. 2(b) shows numerically calculated values of $a_R/\mathrm{T}^{\,1/6}$ plotted as a function of $\log_{10}\mathrm{T}$ and verifies that $a_R/\mathrm{T}^{1/6} \approx 0.9$, as $\mathrm{T}\to\infty$.

Case II When surface tension is sole agency causing instability

On substituting $\Gamma = M/R$ on left hand side of the eigen value Eqn. (3.5), we find that

$$\frac{M}{48}+\frac{R}{320}=\frac{1}{(1-\alpha_2T_0)}\left\{1+\frac{a^2}{15}\left[\frac{\left(\Gamma+\frac{1}{8}\right)^2+\frac{1}{448}}{\left(\Gamma+\frac{1}{12}\right)^2+\frac{1}{180}}\right]+\frac{a^4}{420}\left[\frac{\left(\Gamma+\frac{7}{48}\right)^2+\frac{5}{6912}}{\left(\Gamma+\frac{1}{12}\right)^2+\frac{1}{180}}\right]+\frac{\mathrm{T}}{240}\left[\frac{\left(\Gamma+\frac{1}{6}\right)^2}{\left(5+2a^2\right)\left[\left(\Gamma+\frac{1}{12}\right)^2+\frac{1}{180}\right]}\right]\right\}. \tag{4.3}$$

When $\Gamma\to\infty$ (or $R\to 0$) implies that in the absence of buoyancy effect, surface tension is the sole agency causing instability. In this case, we obtain M from the eigen value Eqn. (4.3) in terms of a, α_2T_0 and T as

$$M=\frac{48}{(1-\alpha_2T_0)}\left\{1+\frac{a^2}{15}+\frac{a^4}{420}+\frac{\mathrm{T}}{240\left(5+2a^2\right)}\right\}. \tag{4.4}$$

The numerical values of M_c and a_M computed using the relation (4.4), for various values of α_2T_0 and T are presented in Table 2. When $\mathrm{T} = 0$, from Table 2 we find that values of both M_c and a_M, for various values of α_2T_0 agree precisely with corresponding values obtained by Gupta and Surya [9].

Table 2 Values of M_c and a_M for various values of T when α_2T_0= 0, 0.3 and 0.5.

T	$\alpha_2T_0=0$		$\alpha_2T_0=0.3$		$\alpha_2T_0=0.5$	
	M_c	a_M	M_c	a_M	M_c	a_M
0	48.00	0.00	68.57	0.00	96.00	0.00
1	48.04	0.00	68.63	0.00	96.08	0.00
10	48.40	0.00	69.14	0.00	96.80	0.00
100	52.00	0.00	74.29	0.00	104.00	0.00
200	56.00	0.00	80.00	0.00	112.00	0.00
250	57.90	0.52	82.71	0.52	115.80	0.52
500	65.51	1.13	93.59	1.13	131.02	1.13
10^3	76.70	1.62	109.57	1.62	153.40	1.62
10^4	171.08	3.28	244.40	3.28	342.16	3.28
10^6	2211.29	8.35	3158.99	8.35	4422.58	8.35
10^8	43531.40	18.60	62187.8	18.60	87062.9	18.60

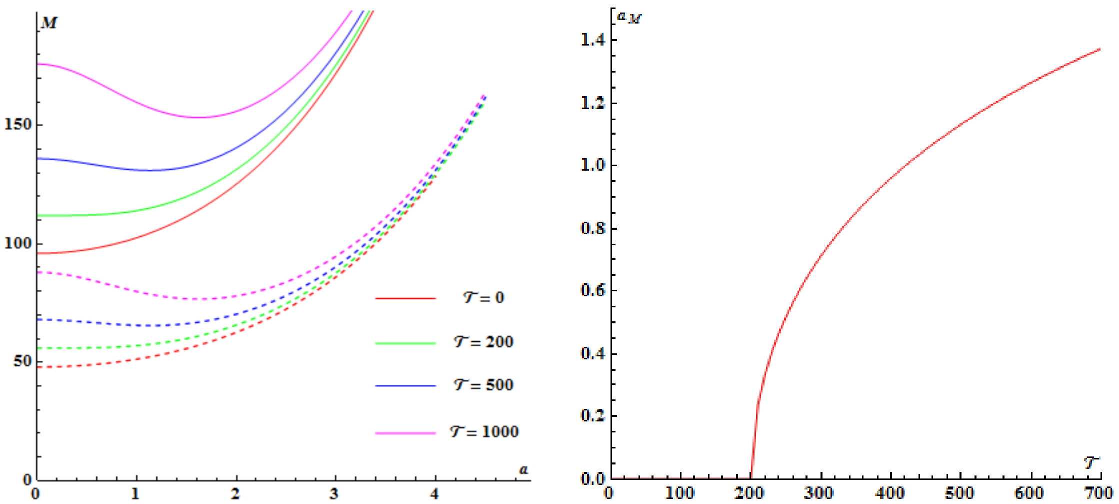
For a prescribed value of $\alpha_2 T_0$, an increase in the value of $\mathcal{T}$ leads to increased values of both M_c and a_M, indicating that the rotation has stabilizing effect with formation of cells of decreased sizes on the onset of surface tension driven convection. On the other hand, for a prescribed value of $\mathcal{T}$ increase in the value of $\alpha_2 T_0$ leads to a greater value of M_c indicating that hotter the liquid layer more the postponement of the onset of instability. It is interesting to note that value of the critical wave number a_M remains unchanged for various values of $\alpha_2 T_0$.

Here it is pointed out that Friedrich and Rudraiah [11] made a small but significant error while solving the eighth order boundary value problem using six boundary conditions without making use of the boundary conditions on vorticity (in terms of velocity profile) and choosing the velocity trial function as $w = z^2(1-z)$ [their Eqn. (24)], a third order polynomial satisfying three boundary conditions which makes no contribution to the term $Dw(1)D^4w(1)$ in the eigen value equation [their Eqn. (23)]. A comparison between the corresponding values of M_c and a_M for various values of $\mathcal{T}$ obtained inaccurately by Friedrich and Rudraiah [11] for the basic linear temperature profile and by us (when $\alpha_2 T_0 = 0$) are given in the Table 3.

Table 3 Comparison with numerical results of Friedrich and Rudraiah[11].

$\mathcal{T}$	Present analysis		Friedrich and Rudraiah[11] (for f(z) = 1)	
	M_c	a_M	M_c	a_M
0	48.00	0.00	48.00	0.00
1	48.04	0.00	48.26	0.29
10	48.40	0.00	50.27	0.83
100	52.00	0.00	61.54	1.93
10^3	76.70	1.62	107.14	3.57

The (a, M) curves corresponding to neutral stability are plotted in Fig. 3(a), using the relation (4.4) for various prescribed values of $\mathcal{T}$ when $\alpha_2 T_0 = 0$ (dotted curves) and $\alpha_2 T_0 = 0.5$ (thick curves), which shows that the rotation has stabilizing effect on the onset of surface tension driven convection. Further, Fig. 3(a) also illustrates that a relatively hotter layer of liquid is more stable than the cooler one under the effect of rotation. The variation of the critical wave number a_M with $\mathcal{T}$ is illustrated in Fig. 3(b).



(a) (b)

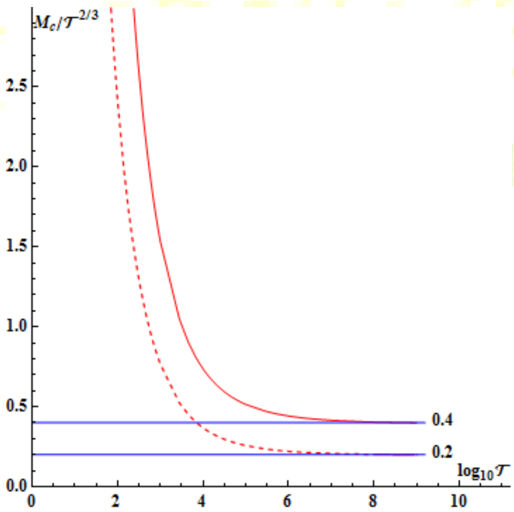
Figure. 3. (a) Neutral stability curves at the onset of Bènard-Marangoni convection for various values of T when $\alpha_2 T_0$ = 0 (dotted curves) $\alpha_2 T_0$ = 0.5 (thick curves). (b) Variation of a_M as a function of T.

The numerical calculations indicate that the marginal stability curves have minimum of M at $a_M = 0$ up to a certain value of the Taylor number T (≤ 200 approximately) and that M_c occurs at a non-zero value of a_M when T is greater. In this case, we note that at the onset of convection to occur at a non zero wave number the threshold speed of rotation ($\mathrm{T} \approx 200$) is relatively more than that ($\mathrm{T} \approx 130$) obtained for the case of purely buoyancy driven convection discussed above.

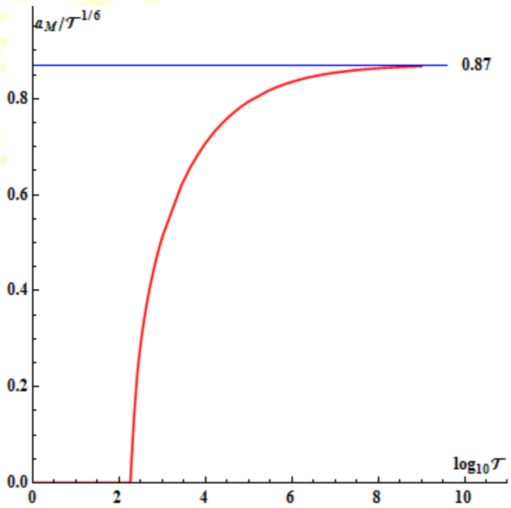
The asymptotic behavior of M_c and a_M obtained numerically for large value of the Taylor number T are given as

$$M_c \approx \frac{0.2}{(1-\alpha_2 T_0)}(\mathrm{T})^{\frac{2}{3}} \text{ and } a_M \approx 0.87(\mathrm{T})^{\frac{1}{6}}. \tag{4.5}$$

We find that the asymptotic behavior of M_c crucially depends on both T and $\alpha_2 T_0$, whereas the asymptotic behavior of a_M only depends on T and is independent of $\alpha_2 T_0$. Fig. 4(a) shows numerically calculated values of $M_c / \mathrm{T}^{2/3}$ plotted as a function of $\log_{10} \mathrm{T}$ and verifies that $M_c / \mathrm{T}^{2/3} \approx 0.2$ and 0.4 respectively when $\alpha_2 T_0$ = 0 (dotted curve) and $\alpha_2 T_0$ = 0.5 (thick curve), as $\mathrm{T} \to \infty$. Fig. 4(b) shows numerically calculated values of $a_M / \mathrm{T}^{1/6}$ plotted as a function of $\log_{10} \mathrm{T}$ and verifies that $a_M / \mathrm{T}^{1/6} \approx 0.87$, as $\mathrm{T} \to \infty$.



(a)



(b)

Figure. 4 Asymptotic results in the limit $\mathrm{T} \to \infty$ plotted as function of $\log_{10} \mathrm{T}$ (a) $M_c / \mathrm{T}^{2/3}$ when $\alpha_2 T_0$ = 0 (dotted curve) $\alpha_2 T_0$ = 0.5 (thick curve). (b) $a_M / \mathrm{T}^{1/6}$.

Case III when both buoyancy and surface tension cause instability

The numerical values of R_c and corresponding wave number a_R, computed with the aid of the eigen value Eqn. (3.5), for various prescribed values of Γ and T are presented in Table 4 and Table 5 when $\alpha_2 T_0$ = 0 and 0.5 respectively. For a fixed value of T, both Table 4 and 5 show that R_c decreases with increase in Γ (or decrease in depth d of the liquid layer). In other words, increasing effect of the surface tension causes reduction in the critical Rayleigh number R_c in the presence of rotation, irrespective of whether the layer of liquid is relatively cooler or hotter. Further, from Table 4 and Table 5, we also find that both R_c and a_R increase monotonically with T, for a prescribed value of Γ. These results show that the inhibiting effect of the rotation remains unchanged on the onset of convection driven under the joint action of buoyancy and surface tension, and that convective cells formed at the onset of convection are of relatively smaller in size irrespective of whether the layer of liquid is relatively cooler or hotter. On comparing the results between Table 4 and Table 5, we find that an increase in the value of $\alpha_2 T_0$ leads to an increased value of R_c corresponding to fixed values of Γ and T which means that a relatively hotter layer of liquid is more stable than the cooler one gravitationally, in the presence of surface tension and under the effect of rotation. It may be noted that the critical wave number a_R remains unchanged whether the layer of liquid is relatively hotter or cooler under the identical conditions otherwise.

Table 4 Values of R_c and a_R for various values of Γ and T when $\alpha_2 T_0 = 0$.

T	$\Gamma = 0$		$\Gamma = 0.5$		$\Gamma = 1$		$\Gamma = 10$	
	R_c	a_R	R_c	a_R	R_c	a_R	R_c	a_R
0	320.00	0.00	73.85	0.00	41.74	0.00	4.73	0.00
10	325.93	0.00	74.64	0.00	42.14	0.00	4.77	0.00
100	379.26	0.00	81.75	0.00	45.75	0.00	5.13	0.00
130	397.04	0.11	84.13	0.00	46.96	0.00	5.25	0.00
150	408.45	0.42	85.71	0.00	47.76	0.00	5.33	0.00
200	434.31	0.74	89.61	0.38	49.76	0.29	5.53	0.10
250	457.43	0.93	93.16	0.65	51.60	0.60	5.72	0.53
500	550.91	1.43	107.43	1.22	59.01	1.18	6.48	1.14
10^3	689.95	1.89	128.46	1.70	69.92	1.67	7.60	1.63
10^6	29248.50	8.76	4231.48	8.46	2177.89	8.42	220.98	8.36

Table 5 Values of R_c and a_R for various values of and T when $\alpha_2 T_0 = 0.5$.

T	$\Gamma = 0$		$\Gamma = 0.5$		$\Gamma = 1$		$\Gamma = 10$	
	R_c	a_R	R_c	a_R	R_c	a_R	R_c	a_R
0	640.00	0.00	147.69	0.00	83.48	0.00	9.46	0.00
10	651.85	0.00	149.27	0.00	84.28	0.00	9.54	0.00
100	758.52	0.00	163.51	0.00	91.51	0.00	10.26	0.00
130	794.07	0.11	168.26	0.00	93.92	0.00	10.50	0.00
150	816.90	0.42	171.42	0.00	95.52	0.00	10.66	0.00
200	868.63	0.74	179.22	0.38	99.52	0.29	11.06	0.10
250	914.86	0.93	186.32	0.65	103.21	0.60	11.44	0.53
500	1101.82	1.43	214.85	1.22	118.03	1.18	12.96	1.14
10^3	1379.90	1.89	256.93	1.70	139.85	1.67	15.19	1.63
10^6	58496.90	8.76	8462.96	8.46	4355.78	8.42	441.95	8.36

The variation of the critical Rayleigh number R_c with T for various prescribed values of Γ when $\alpha_2 T_0 = 0$ and 0.5 are illustrated in Fig. 5(a) and Fig. 5(b) respectively. Fig. 5(c) illustrates the variation of a_R with T for various prescribed values of Γ. Note that values of a_R remain unchanged with respect to $\alpha_2 T_0$. Further, numerical calculations indicate that the marginal stability curves have minimum of R at $a_R = 0$ up to a certain threshold value of the Taylor number T which depends upon Γ. For instance, the marginal stability curves plotted in Fig. 5(d) corresponding to the case $\Gamma = 1$ in which buoyancy and surface tension equally act together and when $\alpha_2 T_0 = 0$ (dotted curves) and $\alpha_2 T_0 = 0.5$ (thick curves) illustrates that R attains its minimum at $a_R = 0$ up to a certain threshold value of the Taylor number $\mathrm{T} \approx 187$. These results indicate that the layer of liquid does not split up into more than single cell (largest size) up to threshold speed of the rotation. An increase in the value of Γ from 0 onwards means that role inclusion of the effect of surface.

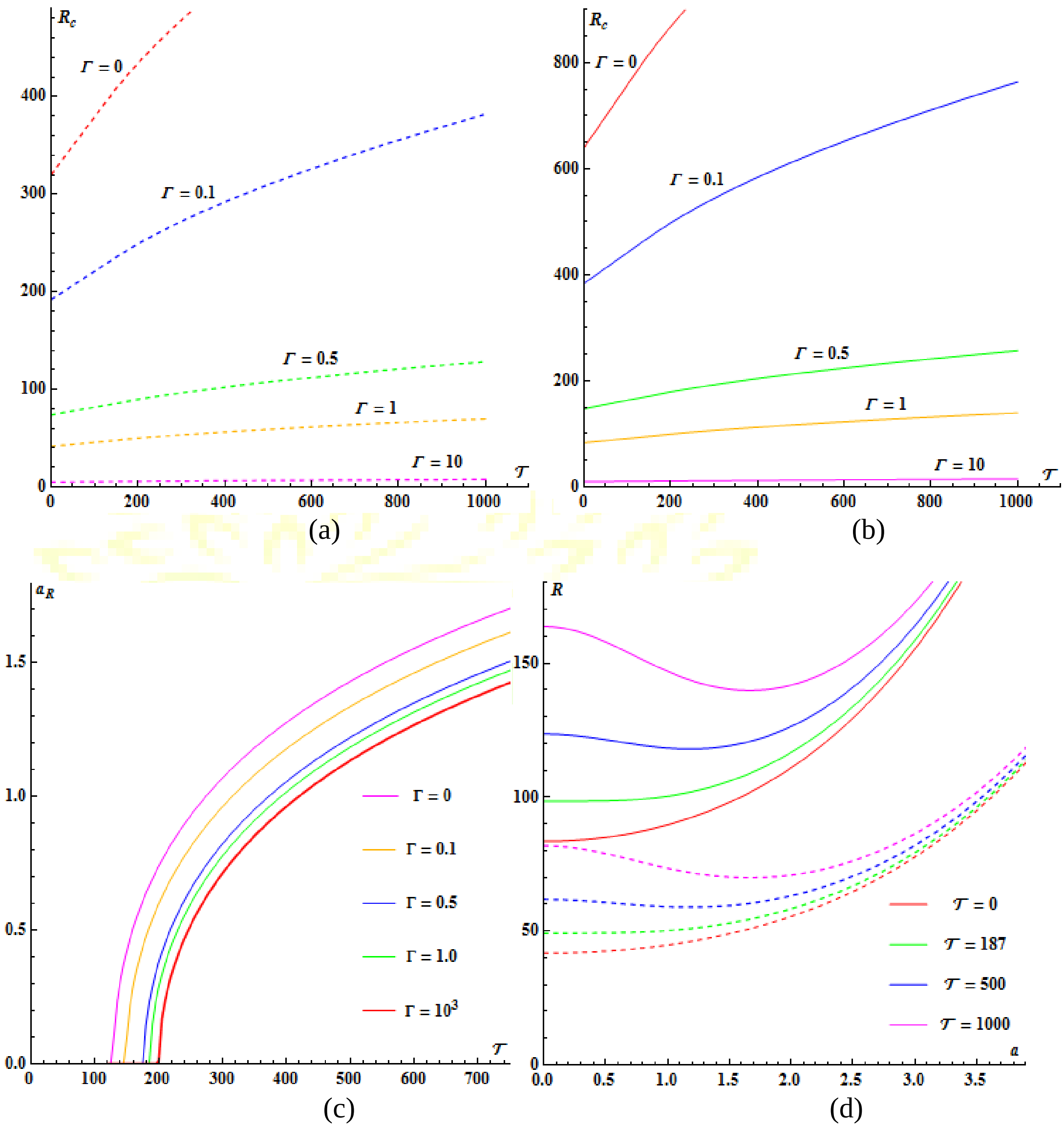


Figure. 5 (a) Variation of R_c with T for various values of Γ when $\alpha_2 T_0 = 0$.
(b) Variation of R_c with T for various values of Γ when $\alpha_2 T_0 = 0.5$. (c) Variation of a_R with T for various values of Γ (d) Variation of R with a for various values of T .

For more clarity, we now discuss the results in terms of the usual parameters R and M when both buoyancy and surface tension effects are present. The neutral stability condition (3.5) may then be put in the form as

$$\frac{R}{R_c}+\frac{M}{M_c}=\frac{1}{(1-\alpha_2 T_0)}\left\{1+\frac{a^2}{15}\left[\frac{\left(\Gamma+\frac{1}{8}\right)^2+\frac{1}{448}}{\left(\Gamma+\frac{1}{12}\right)^2+\frac{1}{180}}\right]+\frac{a^4}{420}\left[\frac{\left(\Gamma+\frac{7}{48}\right)^2+\frac{5}{6912}}{\left(\Gamma+\frac{1}{12}\right)^2+\frac{1}{180}}\right]\right.$$
$$\left.+\frac{\mathrm{T}}{240}\left[\frac{\left(\Gamma+\frac{1}{6}\right)^2}{\left(5+2a^2\right)\left[\left(\Gamma+\frac{1}{12}\right)^2+\frac{1}{180}\right]}\right]\right\}. \tag{4.6}$$

Where R_c = 320 is the critical value of the Rayleigh number in the absence of surface tension effect (when $\alpha_2 T_0$ = 0 and T = 0) and M_c = 48 is the critical value of the Marangoni number in the absence of buoyancy effect (when $\alpha_2 T_0$ = 0 and T = 0). The numerical values of M/M_c (normalized) and corresponding critical wave number can be computed using (4.6), for prescribed normalized values R/R_c in which value of R equals its critical value corresponding to given Γ, T and $\alpha_2 T_0$. The normalized values of R, M and corresponding wave number for various values of T when $\alpha_2 T_0$ = 0 and 0.5, are tabulated in Table 6 and Table 7 respectively. For a fixed value of T, we observe from Table 6 and Table 7 that M/M_c increases with decrease in R/R_c, indicating that the two agencies causing instability reinforce each other, irrespective of whether the layer of liquid is relatively cooler or hotter.

The (R,M)-loci corresponding to neutral stability curves for the combined surface tension and buoyancy effects, normalized for critical values R_c and M_c for various values of T when $\alpha_2 T_0$= 0 and 0.5, are plotted in Fig. 6(a).

Table 6 Normalized R and M for various values of Γ and T when $\alpha_2 T_0$ = 0.

Γ	T = 0			T = 10^3			T = 10^6		
	R/R_c	M/M_c	a_c	R/R_c	M/M_c	a_c	R/R_c	M/M_c	a_c
0	1.00	0.00	0.00	1.00	0.00	1.89	1.00	0.00	8.76
10^{-3}	0.99	0.01	0.00	0.99	0.01	1.89	0.99	0.01	8.76
10^{-2}	0.94	0.06	0.00	0.93	0.08	1.88	0.93	0.12	8.74
10^{-1}	0.60	0.40	0.00	0.55	0.50	1.80	0.51	0.67	8.62
0.5	0.23	0.77	0.00	0.19	0.84	1.70	0.15	0.96	8.46
1	0.13	0.87	0.00	0.10	0.91	1.67	0.07	0.99	8.42
10	0.02	0.99	0.00	0.01	0.99	1.63	0.01	1.00	8.36
10^2	0.00	1.00	0.00	0.00	1.00	1.62	0.00	1.00	8.35
10^3	0.00	1.00	0.00	0.00	1.00	1.62	0.00	1.00	8.35
10^6	0.00	1.00	0.00	0.00	1.00	1.62	0.00	1.00	8.35

Table 7 Normalized R and M for various values of Γ and T when $\alpha_2 T_0$ = 0.5.

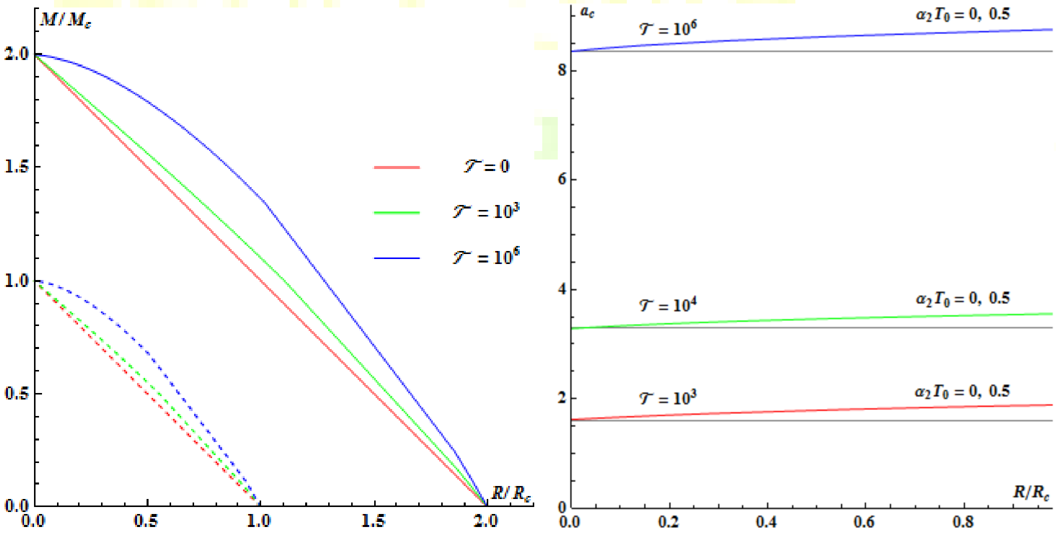
Γ	T = 0			T = 10^3			T = 10^6		
	R/R_c	M/M_c	a_c	R/R_c	M/M_c	a_c	R/R_c	M/M_c	a_c
0	2.00	0.00	0.00	2.00	0.00	1.89	2.00	0.00	8.76
10^{-3}	1.99	0.01	0.00	1.99	0.02	1.89	1.99	0.03	8.76
10^{-2}	1.88	0.13	0.00	1.87	0.17	1.88	1.86	0.25	8.74
10^{-1}	1.20	0.80	0.00	1.11	1.00	1.80	1.02	1.35	8.62
0.5	0.46	1.54	0.00	0.37	1.68	1.70	0.29	1.91	8.46
1	0.26	1.74	0.00	0.20	1.82	1.67	0.15	1.97	8.42

10	0.03	1.97	0.00	0.02	1.98	1.63	0.02	2.00	8.36
10^2	0.00	2.00	0.00	0.00	2.00	1.62	0.00	2.00	8.35
10^3	0.00	2.00	0.00	0.00	2.00	1.62	0.00	2.00	8.35
10^6	0.00	2.00	0.00	0.00	2.00	1.62	0.00	2.00	8.35

The stable states correspond to the region $R < R_c$ and $M < M_c$. When T = 0, the curves corresponding to $\alpha_2 T_0$= 0 (dotted) and 0.5 (thick) in the (R, M) plane as shown in Fig. 6(a) are straight lines represented by

$$\frac{R}{R_c} + \frac{M}{M_c} = \frac{1}{(1-\alpha_2 T_0)} \quad . \tag{4.7}$$

This indicates that there is a maximum reinforcement between the two mechanisms causing instability and the coupling between the two mechanisms is perfect in the absence of rotation and this situation occurs at zero wave number. As T increases the locus goes away from the line (4.7), showing that the coupling between the two mechanisms causing instability remains no longer perfect and coupling between them becomes less tight. Further, Fig. 6(a) also illustrates that the rotation suppresses convection more effectively in a relatively hotter layer of liquid that the cooler one. Fig. 6(b) illustrates the variation of the critical wave number corresponding to the marginal stability for prescribed values of the Taylor number T , indicating that size of convective cells increases at the onset of convection. It may be noted that the variation of the critical wave number is independent of $\alpha_2 T_0$



(a) (b)

Figure. 6 a)Variation of normalized Marangoni and Rayleigh numbers for various values of T when $\alpha_2 T_0$ =0 (dotted curves) and $\alpha_2 T_0$ = 0.5 (thick curves). b) Variation of wave numbers versus normalized Rayleigh numbers for various values of T when $\alpha_2 T_0$ = 0 and 0.5.

5. Conclusions

The problem of the onset of buoyancy and surface tension driven thermal convection in a horizontal liquid layer heated from below rotating uniformly about a vertical axis has been studied theoretically, using the modified linear stability analysis. We conclude that

I. The increase in speed of rotation always has stabilizing effect on the onset of convection in a relatively hotter or cooler liquid layer, irrespective of whether the two mechanisms causing instability act individually or simultaneously. It is interesting to note that the critical wave number at the onset of combined surface tension and buoyancy driven convection is found to be zero up to a certain threshold speed of rotation (depending upon Γ) which becomes non-zero when speed of rotation is greater than the threshold.

II. For large values of the Taylor number, the asymptotic behavior of the critical Rayleigh number (in the absence of surface tension) as well as the critical Marangoni number (in the absence of buoyancy) were found to be significantly dependent on whether the layer of liquid is relatively hotter or cooler.

III. The two mechanisms causing instability reinforce each other and are perfectly coupled in the absence of rotation. However, for large values of the Taylor number, the coupling between the two mechanisms causing instability remains no longer perfect and it becomes less tight.

IV. The uniform rotation suppresses convection more effectively in a relatively hotter layer of liquid than the cooler one.

Experimental work in the laboratory would be welcomed in order to check the qualitative as well as quantitative changes brought in this theoretical study.

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